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Synchronization waves in arrays of driven chaotic systems

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Chaotic synchronization has been observed experimentally and numerically in arrays of Chua's circuits, arranged in both linear and ring geometries, that are coupled by using the method recently introduced by Güémez and Matías [Phys. Rev. E **52**, R2145 (1995)]. For open linear geometries, the chaotic cells are seen to synchronize consecutively as a *synchronization wave* spreads through the array. Instead, for circular loops it is found that there is a critical number of cells above which the uniform synchronized state is not stable. [S1063-651X(96)51210-9]

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Synchronization among coupled nonlinear oscillators is pervasive in nature (see, e.g., Ref. [1]). Although counterintuitive at first sight, synchronization has been also found for some coupled chaotic systems [2,3] both when using linear (or diffusive) coupling [2,4–6] and through driving [3,7]. These results give support to the point of view that chaotic behavior may even be advantageous in some situations. In this context, chaos can be regarded as a reservoir of periodic behavior that can be activated in response to changing stimuli or external conditions [8].

In the present paper, we shall consider arrays of chaotic systems that are coupled through driving. In particular, we shall resort to a variant of the Pecora-Carroll method [3] recently introduced in Ref. [7]. The main advantage of this method is that the dynamical evolution of the driving signal in the response is not suppressed and thus one can consider more general arrangements of the connected systems, such as cascades [9]. We have applied this coupling method to the case of one-dimensional arrays of Chua's circuits in the chaotic regime, considering two different topologies, linear and in a loop, quite exhaustively, both through numerical simulation and experimentally. This method might allow me to design in a more systematic and compact way circuits that can be employed as chaotic filters for secure communications [9,10] or as cells potentially useful for arrays of cellular neural networks (CNN) [11].

The basic unit (cell) of our study is Chua's circuit, an electronic oscillator that has been shown to exhibit a variety of bifurcation and chaotic phenomena [12]. The dynamics of an array with N units can be modeled by a system of $3N$ first-order autonomous nonlinear differential equations that in explicit rescaled dimensionless form (including coupling) are written as

$$\left.\begin{aligned} k\bar{x}_k &= \alpha[y_k - x_k - f(\overline{x_k})] \\ \dot{y}_k &= x_k - y_k + z_k \\ \dot{z}_k &= -\beta y_k - \gamma z_k \end{aligned}\right\} \quad k=1,\ldots,N, \tag{1}$$

where $\alpha = C_2/C_1$, $\beta = C_2/(LG^2)$, and $\gamma = (C_2 r_0)/(LG)$. The three-segment piecewise-linear characteristic of the nonlinear resistor (Chua's diode) is given by

$$f(x) = \left\{ bx + \frac{1}{2}(a-b)[|x+1| - |x-1|] \right\}, \tag{2}$$

where a and b are the slopes of the inner and outer regions, respectively, of $f(x)$. Driving is introduced through $f(\overline{x_k})$ and $\overline{x_k} = x_{k-1}$ for $k \neq 1$, while for $k=1$ and depending on the type of arrangement, one has $\overline{x_1} = x_1$ for linear arrays and for closed loops $\overline{x_1} = x_N$.

An experimental setup formed by four Chua circuits whose components are defined by $(C_1, C_2, L, r_0, R) = (12$ nF, 100 nF, 10 mH, 9 Ω, 1 kΩ), has been built to implement Eqs. (1). The tolerances of the components employed are 10% for inductances, 5% for capacitances, and 1% for resistances. The experiment has been designed in such a way that one can connect the individual circuits in a variety of ways. The circuits were sampled with a digital oscilloscope (Hewlett-Packard 54601) with a maximum sample rate of 20 million samples per sec, with 8-bit A/D resolution, and a record length of 4000 points. Figure 1 shows the experimental de-

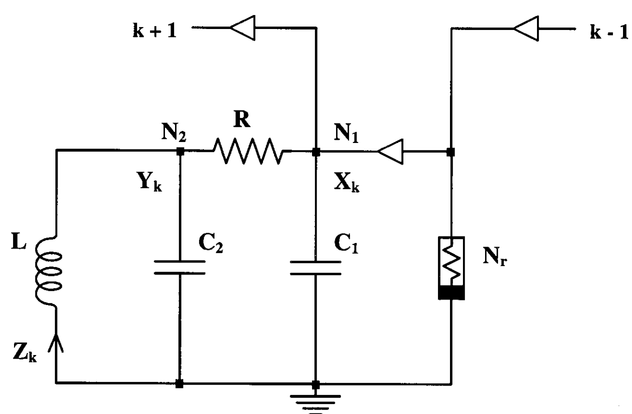

FIG. 1. Schematic diagram of an experimental Chua circuit coupled with its neighbors through the nonlinear element.

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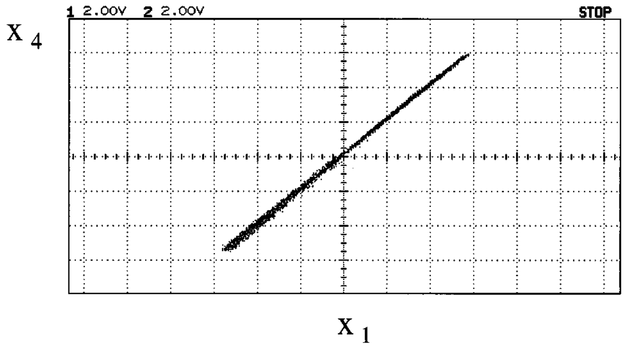

FIG. 2. Representation of the voltage through capacitor C_1 corresponding to the fourth circuit, x_4, vs the same quantity for the first circuit, x_1, for an experimental setting of four circuits connected linearly; see Eq. (1). This plot indicates synchronization among the circuits.

sign of one Chua circuit coupled with its neighbors according to Eq. (1) of the system.

The following set of parameters has been used for the numerical calculations throughout the present contribution: $(\alpha,\beta,\gamma,a,b)=(8.23,10,0.09,-1.10,-0.67)$, since they correspond to the components used in the experimental circuit. The set of equations (1), together with the suitable boundary conditions, zero flux for linear arrays and periodic for closed-loop arrays (or rings), have been integrated by using a stable fixed-step fourth-order Runge-Kutta method with a step size of $\Delta t=0.001$ time units (t.u.) (1 t.u.=0.1 μs). In all numerical and experimental simulations shown here, the isolated systems have been allowed to evolve without coupling for an amount of time such that the system dynamics takes place in the double-scroll (chaotic) attractor. Then the circuits are connected with a certain topology in the form described by Eq. (1).

The results obtained in this work, through both numerical simulations and experiments, are different depending on whether one considers linear arrays or closed loops. In the first case, and after some transient time, the array attains a uniform synchronized chaotic state. Thus, Fig. 2 presents the results corresponding to the potential difference along capacitor C_1 (that in the dimensionless form of the circuit corresponds to x) for the last circuit, x_4, versus the same variable for the first circuit, x_1, obtained in the experiment for a linear array of four elements. Synchronization is expressed by the straight line $x_4=x_1$. It is interesting to note that synchronization is quite robust along the array. This feature is not obvious at all, because one could expect that the tolerances inherent in electronic components would cause this phenomenon to deteriorate along the array.

Synchronization does not happen instantaneously, but, instead, takes place as a *synchronization wave* spreads through the system. Figure 3 shows the difference in voltages Δx along capacitor C_1 (taken in node N_1, see Fig. 1) between consecutive circuits ($\Delta x_k=|x_k-x_{k-1}|$) as a function of time and the number of circuits in a linear array. Note that two consecutive circuits synchronize after a given time has passed (this happens when $\Delta x\rightarrow 0$), thus defining the synchronization wave. This wave can be characterized by a constant velocity V_s. Contrary to the case of classical waves in linear systems or autowaves in dissipative media [13], chaotic synchronization waves may carry *information* through the array of cells. This information is encoded through changes in the initial conditions at unit $k=1$, which will define different outputs at the opposite side of the array, $k=N$, for a time longer than N/V_s. Thus, the study of this synchronization wave is very interesting for the comprehension of the transmission of signals in CNNs, which can be used for signal processing.

For linear arrays, it is found that the velocity of this synchronization wave depends essentially on the highest transverse Lyapunov exponent characterizing two coupled systems. The transverse Lyapunov spectrum, a generalization

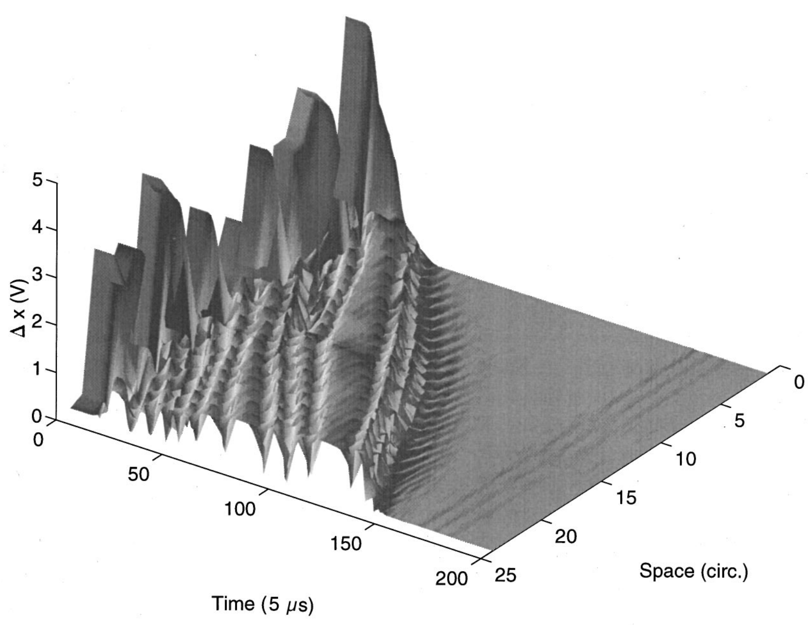

FIG. 3. Spatio-temporal pattern of the differences between the signals of the contiguous circuits in an open linear array consisting of 25 circuits. The synchronization wave is defined by the condition $\Delta x\rightarrow 0$.

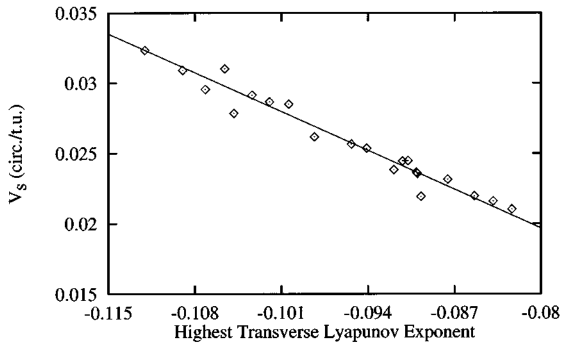

FIG. 4. Representation of the mean velocity of synchronization V_s vs the highest transverse Lyapunov exponent, showing the linear relationship between these two quantities.

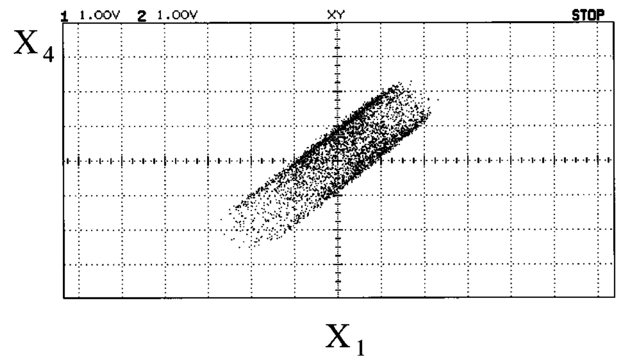

FIG. 5. Representation of the voltage across capacitor C_1 for the fourth circuit, x_4, vs the same quantity for the first circuit, x_1, for an experimental setting of four Chua circuits coupled through the nonlinear element Eq. (1) in a ring geometry.

[6] of the concept of conditional exponents, is obtained for the connection given in Eq. (1) by taking the real part of the eigenvalues of the matrix [7],

$$\mathbf{Z}=\begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \tag{3}$$

where one does not need to perform the usual $t\to\infty$ limit, since the coefficients of the matrix are constant for this particular arrangement. Note that this matrix is obtained [7] by setting to zero the entries corresponding to the connection in the linearized approximation to the flow (the connection enters into the nonlinear element $f(x)$ in Eq. (1), and this is the reason why the matrix (3) has only constant coefficients).

If one recalls that small deviations from the synchronized state behave in the form $\dot{\mathbf{x}}=\mathbf{Z}\mathbf{x}$, this implies that $\mathbf{x}(t)=\mathbf{x}(0)\exp(\mathbf{\Lambda}t)$, where $\mathbf{\Lambda}$ is obtained by transforming matrix $\mathbf{Z}$ to diagonal form. One may assume that this expression is dominated by the first term, i.e., $\mathbf{x}(t)\sim\mathbf{x}(0)\exp(-|\lambda_1|t)$, which defines a relaxational process in a time scale $\tau=1/|\lambda_1|$. Synchronization takes place between contiguous circuits and, since space is discrete with constant spacing, this implies that the velocity of synchronization is of the order of the inverse of the time needed for a single circuit to synchronize, τ, i.e., of the order of $1/|\lambda_1|$. This implies the linear relationship $V_s\sim 1/\tau\sim|\lambda_1|$. Figure 4 shows a graphical representation of V_s versus λ_1 that allows one to verify this linear relationship.

In the case of closed loops, the behavior is more complex due to the presence of feedback. However, in this case stability can be analyzed just based on theoretical grounds, extending slightly the approach of Ref. [6]. The idea is to write the linearized evolution equations for small deviations around the synchronized state (this yields a $3N\times 3N$ problem for N coupled Chua circuits). This matrix will be almost block diagonal, since it will have N blocks with the form of Eq. (3), but will have some sparse off-diagonal elements corresponding to the interaction of the kth node with the $(k-1)$th due to the coupling term.

The fact that the first node is coupled to the last one implies that the structure of this matrix is circulant and so it can be formally decoupled by means of the discrete Fourier transform technique to yield a matrix that has N blocks with dimension 3×3. These N block matrices control the stability of the N Fourier modes corresponding to small deviations about the synchronized state. The first (uniform) mode $k=0$ is the synchronized state itself, and this matrix is the same as that of an isolated Chua circuit. The other $N-1$ matrices allow one to determine the stability of the synchronization manifold against transverse perturbations [6] in terms of the corresponding transverse Lyapunov exponents, since, in principle, the problem is nonlinear. These matrices have the form [14]

$$\mathbf{C}^{(k)}=\begin{pmatrix} -\alpha[1+f'(x)\exp(i\,2\pi k/N)] & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \tag{4}$$

where the index k runs as $k=1,\ldots,(N-1)$. It can be shown that $\mathbf{C}^{(1)}$ is the most relevant matrix in determining the stability of the uniform synchronized state.

The present derivation, but also numerical simulations and the experiments performed with the electronic setup, agree in that the uniform synchronized state is stable when the number of units N in the ring is low enough, until for a certain size N_c the synchronized state loses stability. This number N_c is 5 for a ring of identical Chua circuits with the specifications used in the present work, both by finding the Lyapunov spectrum of $\mathbf{C}^{(1)}$ and by numerical simulation. Instead, experimentally one finds that $N_c=4$ in the coupled electronic circuits (see, e.g., Fig. 5), which can be explained by performing realistic simulations that include the tolerances in the electronic components. The behavior found in Fig. 5 is characterized by delayed synchronization between contiguous circuits that differ by a nonconstant time shift.

In conclusion, we have presented experimental results, supplemented by numerical simulations and some analytical reasoning, about the behavior of small networks of Chua circuits in the chaotic regime, coupled through synchronizing unidirectional connections. For linear arrays, one observes a synchronization wave that propagates throughout the system. It is argued that such a wave may be useful for information transmission or processing, as one is able to send pulses of information in a finite time between the two extremes of the array such that one encodes through the initial conditions of the system. The stability of rings has been also studied and discussed.

Regarding possible future extensions of the present work, one may mention the study of parallel fibers [15], but now composed out of chaotic units, that might be relevant in the study of the behavior of neuronal assemblies. One could also study two-dimensional networks formed by chaotic units that would offer a discrete representation of spatially extended systems with this type of dynamics. In the case in which the local dynamics is excitable, some studies have already revealed many interesting spatiotemporal phenomena [16]. These arrays can also be regarded as chaotic cellular neural networks [11], which could be useful in some computational tasks. This kind of model can also be considered as a simplified representation of neuron assemblies, incorporating the unidirectional character of the connections as found in reality. The symptoms of chaotic behavior in the brain [17] and the important role of synchronization in perceptive processes in mammals [18] indicate that the role of chaos may be constructive in these cases [8].

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